

Bipartite graphs related to mutually disjoint S-permutation matrices

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Abstract

Some numerical characteristics of bipartite graphs in relation to the problem of finding all disjoint pairs of S-permutation matrices in the general $n^2 \times n^2$ case are discussed in this paper. All bipartite graphs of the type $g = \langle R_g \cup C_g, E_g \rangle$, where $|R_g| = |C_g| = 2$ or $|R_g| = |C_g| = 3$ are provided. The cardinality of the sets of mutually disjoint S-permutation matrices in both the 4×4 and 9×9 cases are calculated.

Keyword: *Bipartite graph, Binary matrix, S-permutation matrix, Disjoint matrices, Sudoku*

MSC[2010] code: 05C30, 05B20, 05C50

1 Introduction

Let m be a positive integer. By $[m]$ we denote the set

$$[m] = \{1, 2, \dots, m\}.$$

We let \mathcal{S}_m denote the symmetric group of order m i.e., the group of all one-to-one mappings of the set $[m]$ to itself. If $x \in [m]$, $\rho \in \mathcal{S}_m$, then the image of the element x in the mapping ρ we will denote by $\rho(x)$.

A *bipartite graph* is an ordered triple

$$g = \langle R_g, C_g, E_g \rangle,$$

where R_g and C_g are non-empty sets such that $R_g \cap C_g = \emptyset$. The elements of $R_g \cup C_g$ will be called *vertices*. The set of *edges* is $E_g \subseteq R_g \times C_g = \{\langle r, c \rangle \mid r \in R_g, c \in C_g\}$. Multiple edges are not allowed in our considerations.

The subject of the present work is bipartite graphs considered up to isomorphism.

We refer to [3] or [6] for more details on graph theory.

Let n and k be two nonnegative integers and let $0 \leq k \leq n^2$. We denote by $\mathfrak{G}_{n,k}$ the set of all bipartite graphs of the type $g = \langle R_g, C_g, E_g \rangle$, considered up to isomorphism, such that $|R_g| = |C_g| = n$ and $|E_g| = k$.

Let P_{ij} , $1 \leq i, j \leq n$, be n^2 square $n \times n$ matrices, whose entries are elements of the set $[n^2] = \{1, 2, \dots, n^2\}$. The $n^2 \times n^2$ matrix

$$P = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{21} & P_{22} & \cdots & P_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ P_{n1} & P_{n2} & \cdots & P_{nn} \end{bmatrix}$$

is called a *Sudoku matrix*, if every row, every column and every submatrix P_{ij} , $1 \leq i, j \leq n$ comprise a permutation of the elements of set $[n^2]$, i.e., every number $s \in \{1, 2, \dots, n^2\}$ is found just once in each row, column, and submatrix P_{ij} . Submatrices P_{ij} are called *blocks* of P .

Sudoku is a very popular game and Sudoku matrices are special cases of Latin squares in the class of gerechte designs [1].

A matrix is called *binary* if all of its elements are equal to 0 or 1. A square binary matrix is called *permutation matrix*, if in every row and every column there is just one 1.

Let us denote by Σ_{n^2} the set of all $n^2 \times n^2$ permutation matrices of the following type:

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix},$$

where for every $s, t \in \{1, 2, \dots, n\}$, A_{st} is a square $n \times n$ binary submatrix (block) with only one element equal to 1.

The elements of Σ_{n^2} will be called *S-permutation matrices*.

Two Σ_{n^2} matrices $A = (a_{ij})$ and $B = (b_{ij})$, $1 \leq i, j \leq n^2$ will be called *disjoint*, if there are not elements a_{ij} and b_{ij} with the same indices such that $a_{ij} = b_{ij} = 1$.

The concept of S-permutation matrix was introduced by Geir Dahl [2] in relation to the popular Sudoku puzzle.

Obviously, a square $n^2 \times n^2$ matrix P with entries from $[n^2] = \{1, 2, \dots, n^2\}$ is a Sudoku matrix if and only if there are Σ_{n^2} matrices A_1, A_2, \dots, A_{n^2} , pairwise disjoint, such that P can be written in the following way:

$$P = 1 \cdot A_1 + 2 \cdot A_2 + \cdots + n^2 \cdot A_{n^2} \quad (1)$$

In [5] Roberto Fontana offers an algorithm which returns a random family of $n^2 \times n^2$ mutually disjoint S-permutation matrices, where $n = 2, 3$. For $n = 3$, he ran the algorithm 1000 times and found 105 different families of nine mutually disjoint S-permutation matrices. Then, applying (1), he decided that there are at least $9! \cdot 105 = 38\,102\,400$ Sudoku matrices. This number is very small compared with the exact number of 9×9 Sudoku matrices. In [4] it was shown that there are exactly

$$9! \cdot 72^2 \cdot 2^7 \cdot 27\,704\,267\,971 = 6\,670\,903\,752\,021\,072\,936\,960$$

number of 9×9 Sudoku matrices.

To evaluate the effectiveness of Fontana's algorithm, it is necessary to calculate the probability of two randomly generated matrices being disjoint. As is proved in [2], the number of S-permutation matrices is equal to

$$|\Sigma_{n^2}| = (n!)^{2n}.$$

Thus the question of finding a formula for counting disjoint pairs of S-permutation matrices naturally arises. Such a formula is introduced and verified in [8]. In this paper, we demonstrate this formula to compute the number of disjoint pairs of S-permutation matrices in both the 4×4 and 9×9 cases.

2 A formula for counting disjoint pairs of S-permutation matrices

Let $g = \langle R_g, C_g, E_g \rangle \in \mathfrak{G}_{n,k}$ for some natural numbers n and k and let $v \in V_g = R_g \cup C_g$.

By $N(v)$ we denote the set of all vertices of V_g , adjacent with v , i.e., $u \in N(v)$ if and only if there is an edge in E_g connecting u and v . If v is an isolated vertex (i.e., there is no edge, incident with v), then by definition $N(v) = \emptyset$ and $\text{degree}(v) = |N(v)| = 0$. If $v \in R_g$, then obviously $N(v) \subseteq C_g$, and if $v \in C_g$, then $N(v) \subseteq R_g$.

Let $g = \langle R_g, C_g, E_g \rangle \in \mathfrak{G}_{n,k}$ and let $u, v \in V_g = R_g \cup C_g$. We will say that u and v are equivalent and we will write $u \sim v$ if $N(u) = N(v)$. If u and v are isolated, then by definition $u \sim v$ if and only if u and v belong simultaneously to R_g , or C_g . The above introduced relation is obviously an equivalence relation.

By $V_{g/\sim}$ we denote the obtained factor-set (the set of the equivalence classes) according to relation \sim and let

$$V_{g/\sim} = \{\Delta_1, \Delta_2, \dots, \Delta_s\},$$

where $\Delta_i \subseteq R_g$, or $\Delta_i \subseteq C_g$, $i = 1, 2, \dots, s$, $2 \leq s \leq 2n$. We put

$$\delta_i = |\Delta_i|, \quad 1 \leq \delta_i \leq n, \quad i = 1, 2, \dots, s$$

and for every $g \in \mathfrak{G}_{n,k}$ we define multi-set (set with repetition)

$$[g] = \{\delta_1, \delta_2, \dots, \delta_s\},$$

where $\delta_1, \delta_2, \dots, \delta_s$ are natural numbers, obtained by the above described way.

If $z_1 z_2 \dots z_n$ is a permutation of the elements of the set $[n] = \{1, 2, \dots, n\}$ and we shortly denote ρ this permutation, then in this case we denote by $\rho(i)$ the i -th element of this permutation, i.e., $\rho(i) = z_i$, $i = 1, 2, \dots, n$.

The following theorem is proved in [8]:

Theorem 1 [8] *Let $n \geq 2$ be a positive integer. Then the number D_{n^2} of all disjoint ordered pairs of matrices in Σ_{n^2} is equal to*

$$D_{n^2} = (n!)^{4n} + (n!)^{2(n+1)} \sum_{k=1}^{n^2} (-1)^k \sum_{g \in \mathfrak{G}_{n,k}} \frac{\prod_{v \in R_g \cup C_g} (n - |N(v)|)!}{\prod_{\delta \in [g]} \delta!}. \quad (2)$$

The number d_{n^2} of all non-ordered pairs of disjoint matrices in Σ_{n^2} is equal to

$$d_{n^2} = \frac{1}{2} D_{n^2} \quad (3)$$

□

The proof of Theorem 1 is described in detail in [8] and here we will miss it.

In order to apply Theorem 1 it is necessary to describe all bipartite graphs up to isomorphism $g = \langle R_g, C_g, E_g \rangle$, where $|R_g| = |C_g| = n$.

Let n and k are positive integers and let $g \in \mathfrak{G}_{n,k}$. We examine the ordered $(n+1)$ -tuple

$$\Psi(g) = \langle \psi_0(g), \psi_1(g), \dots, \psi_n(g) \rangle, \quad (4)$$

where $\psi_i(g)$, $i = 0, 1, \dots, n$ is equal to the number of vertices of g incident with exactly i number of edges. It is obvious that $\sum_{i=1}^n i\psi_i(g) = 2k$ is true for all $g \in \mathfrak{G}_{n,k}$. Then formula (2) can be presented

$$D_{n^2} = (n!)^{4n} + (n!)^{2(n+1)} \sum_{k=1}^{n^2} (-1)^k \sum_{g \in \mathfrak{G}_{n,k}} \frac{\prod_{i=0}^n [(n-i)!]^{\psi_i(g)}}{\prod_{\delta \in [g]} \delta!}.$$

Since $(n-n)! = 0! = 1$ and $[n - (n-1)]! = 1! = 1$, then

$$D_{n^2} = (n!)^{4n} + (n!)^{2(n+1)} \sum_{k=1}^{n^2} (-1)^k \sum_{g \in \mathfrak{G}_{n,k}} \frac{\prod_{i=0}^{n-2} [(n-i)!]^{\psi_i(g)}}{\prod_{\delta \in [g]} \delta!}. \quad (5)$$

Consequently, to apply formula (5) for each bipartite graph $g \in \mathfrak{G}_{n,k}$ and for the set $\mathfrak{G}_{n,k}$ of bipartite graphs, it is necessary to obtain the following numerical characteristics:

$$\omega(g) = \frac{\prod_{i=0}^{n-2} [(n-i)!]^{\psi_i(g)}}{\prod_{\delta \in [g]} \delta!} \quad (6)$$

and

$$\theta(n, k) = \sum_{g \in \mathfrak{G}_{n,k}} \omega(g) \quad (7)$$

Using the numerical characteristics (6) and (7), we obtain the following variety of Theorem 1:

Theorem 2

$$D_{n^2} = (n!)^{4n} + (n!)^{2(n+1)} \sum_{k=1}^{n^2} (-1)^k \theta(n, k), \quad (8)$$

where $\theta(n, k)$ is described using formulas (7) and (6).

□

3 Demonstrations in applying of Theorem 2

3.1 Counting the number D_4 of all ordered pairs of disjoint S-permutation matrices for $n = 2$

3.1.1 $k = 1$

In $n = 2$ and $k = 1$, $\mathfrak{G}_{2,1}$ consists of a single graph g_1 shown in Figure 1.

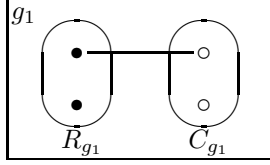


Figure 1: $n = 2, k = 1$

For graph $g_1 \in \mathfrak{G}_{2,1}$ we have:

$$[g_1] = \{1, 1, 1, 1\}$$

$$\Psi(g_1) = \langle \psi_0(g_1), \psi_1(g_1), \psi_2(g_1) \rangle = \langle 2, 2, 0 \rangle$$

Then we get:

$$\omega(g_1) = \frac{[(2-0)!]^2}{1! 1! 1! 1!} = 4$$

and therefore

$$\theta(2, 1) = \sum_{g \in \mathfrak{G}_{2,1}} \omega(g) = 4. \quad (9)$$

3.1.2 $k = 2$

The set $\mathfrak{G}_{2,2}$ consists of three graphs g_2 , g_3 and g_4 depicted in Figure 2.

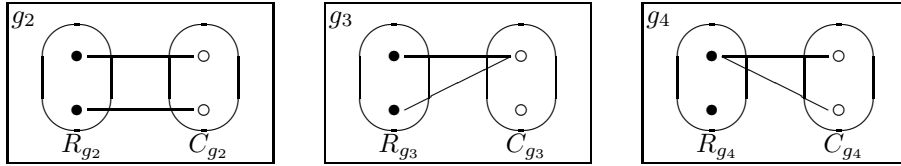


Figure 2: $n = 2, k = 2$

For graph $g_2 \in \mathfrak{G}_{2,2}$ we have:

$$[g_2] = \{1, 1, 1, 1\}$$

$$\Psi(g_2) = \langle \psi_0(g_2), \psi_1(g_2), \psi_2(g_2) \rangle = \langle 0, 4, 0 \rangle$$

$$\omega(g_1) = \frac{[(2-0)!]^0}{1! 1! 1! 1!} = 1$$

For graphs $g_3 \in \mathfrak{G}_{2,2}$ and $g_4 \in \mathfrak{G}_{2,2}$ we have:

$$[g_3] = [g_4] = \{2, 1, 1\}$$

$$\Psi(g_3) = \Psi(g_4) = \langle 1, 2, 1 \rangle$$

$$\omega(g_3) = \omega(g_4) = \frac{[(2-0)!]^1}{2! 1! 1!} = 1$$

Then for the set $\mathfrak{G}_{2,2}$ we get:

$$\theta(2, 2) = \sum_{g \in \mathfrak{G}_{2,2}} \omega(g) = 1 + 1 + 1 = 3. \quad (10)$$

3.1.3 $k = 3$

In $n = 2$ and $k = 3$, $\mathfrak{G}_{2,3}$ consists of a single graph g_5 shown in Figure 3.

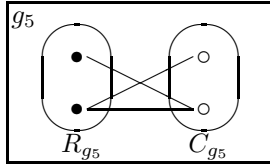


Figure 3: $n = 2, k = 3$

For graph $g_5 \in \mathfrak{G}_{2,3}$ we have:

$$[g_5] = \{1, 1, 1, 1\}$$

$$\Psi(g_5) = \langle \psi_0(g_5), \psi_1(g_5), \psi_2(g_5) \rangle = \langle 0, 2, 2 \rangle$$

Then we get:

$$\omega(g_5) = \frac{[(2-0)!]^0}{1! 1! 1! 1!} = 1$$

and therefore

$$\theta(2, 3) = \sum_{g \in \mathfrak{G}_{2,3}} \omega(g) = 1. \quad (11)$$

3.1.4 $k = 4$

When $n = 2$ and $k = 4$ there is only one graph and this is the complete bipartite graph g_6 which is shown in Figure 4.

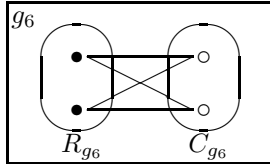


Figure 4: $n = 2, k = 4$

For graph $g_6 \in \mathfrak{G}_{2,4}$ we have:

$$[g_6] = \{2, 2\}$$

$$\Psi(g_6) = \langle \psi_0(g_6), \psi_1(g_6), \psi_2(g_6) \rangle = \langle 0, 0, 4 \rangle$$

Then we get:

$$\omega(g_6) = \frac{[(2-0)!]^0}{2! \ 2!} = \frac{1}{4}$$

and therefore

$$\theta(2, 4) = \sum_{g \in \mathfrak{G}_{2,1}} \omega(g) = \frac{1}{4}. \quad (12)$$

Having in mind the formulas (8), (9), (10), (11) and (12) for the number D_4 of all ordered pairs disjoint S-permutation matrices in $n = 2$ we finally get:

$$\begin{aligned} D_4 &= (2!)^8 + (2!)^6 [-\theta(2, 1) + \theta(2, 2) - \theta(2, 3) + \theta(2, 4)] = \\ &= 256 + 64 \left(-4 + 3 - 1 + \frac{1}{4} \right) = 144. \end{aligned} \quad (13)$$

The number d_4 of all non-ordered pairs disjoint matrices from Σ_4 is equal to

$$d_4 = \frac{1}{2} D_4 = 72. \quad (14)$$

3.2 Counting the number D_9 of all ordered pairs of disjoint S-permutation matrices for $n = 3$

3.2.1 $k = 1$

Graph g_7 , which is displayed in Figure 5 is the only bipartite graph belonging to the set $\mathfrak{G}_{3,1}$.

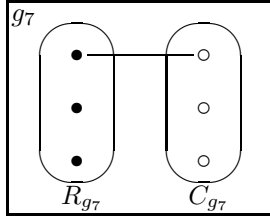


Figure 5: $n = 3, k = 1$

For graph $g_7 \in \mathfrak{G}_{3,1}$ we have:

$$[g_7] = \{1, 1, 2, 2\}$$

$$\Psi(g_7) = \langle \psi_0(g_7), \psi_1(g_7), \psi_2(g_7), \psi_3(g_7), \psi_4(g_8) \rangle = \langle 4, 2, 0, 0 \rangle$$

Then we get:

$$\omega(g_7) = \frac{[(3-0)!]^4 [(3-1)!]^2}{1! \ 1! \ 2! \ 2!} = \frac{6^4 \cdot 2^2}{1 \cdot 1 \cdot 2 \cdot 2} = 1296$$

and therefore

$$\theta(3, 1) = \sum_{g \in \mathfrak{G}_{3,1}} \omega(g) = 1296. \quad (15)$$

3.2.2 $k = 2$

In this case $\mathfrak{G}_{3,2} = \{g_8, g_9, g_{10}\}$. The graphs g_8 , g_9 and g_{10} are shown in Figure 6.

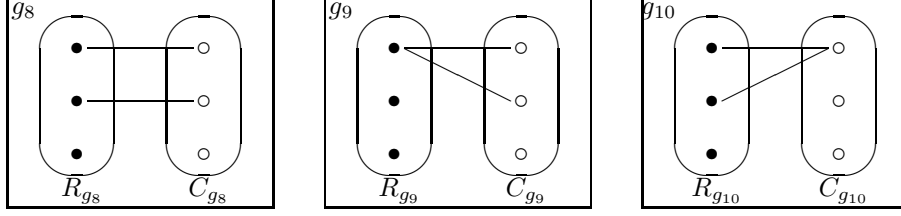


Figure 6: $n = 3, k = 2$

For graph $g_8 \in \mathfrak{G}_{3,2}$ we have:

$$[g_8] = \{1, 1, 1, 1, 1, 1\}$$

$$\Psi(g_8) = \langle \psi_0(g_8), \psi_1(g_8), \psi_2(g_8), \psi_3(g_8), \psi_4(g_8) \rangle = \langle 2, 4, 0, 0 \rangle$$

$$\omega(g_8) = \frac{[(3-0)!]^2 [(3-1)!]^4}{1! 1! 1! 1! 1! 1!} = 6^2 \cdot 2^4 = 576$$

For graphs $g_9 \in \mathfrak{G}_{3,2}$ and $g_{10} \in \mathfrak{G}_{3,2}$ we have:

$$[g_9] = [g_{10}] = \{1, 1, 2, 2\}$$

$$\Psi(g_9) = \Psi(g_{10}) = \langle 3, 2, 1, 0 \rangle$$

$$\omega(g_9) = \omega(g_{10}) = \frac{[(3-0)!]^3 [(3-1)!]^2}{1! 1! 2! 2!} = \frac{6^3 \cdot 2^2}{1 \cdot 1 \cdot 2 \cdot 2} = 216$$

Then for the set $\mathfrak{G}_{3,2}$ we get:

$$\theta(3, 2) = \sum_{g \in \mathfrak{G}_{3,2}} \omega(g) = 576 + 216 + 216 = 1008. \quad (16)$$

3.2.3 $k = 3$

When $n = 3$ and $k = 3$ the set $\mathfrak{G}_{3,3} = \{g_{11}, g_{12}, g_{13}, g_{14}, g_{15}, g_{16}\}$ consists of six bipartite graphs, which are shown in Figure 7.

For graph $g_{11} \in \mathfrak{G}_{3,3}$ we have:

$$[g_{11}] = \{1, 1, 1, 1, 1, 1\}$$

$$\Psi(g_{11}) = \langle 0, 6, 0, 0 \rangle$$

$$\omega(g_{11}) = \frac{[(3-0)!]^0 [(3-1)!]^6}{1! 1! 1! 1! 1! 1!} = 6^0 \cdot 2^6 = 64$$

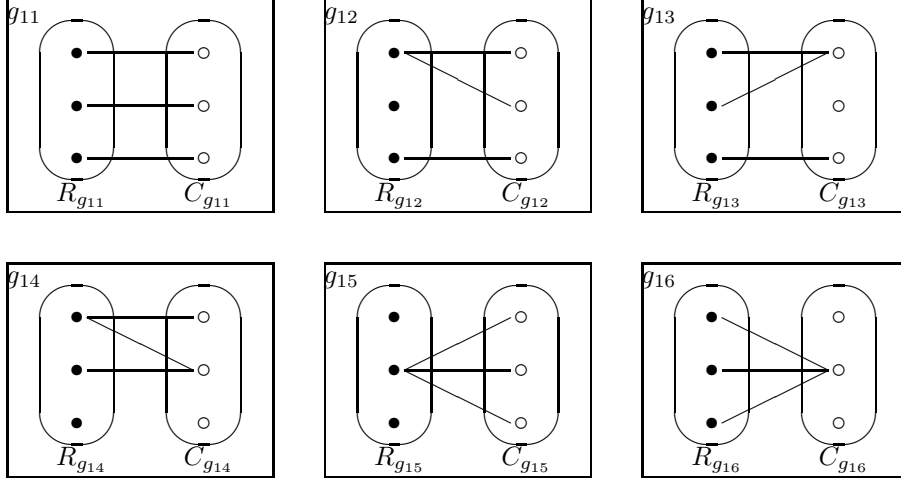


Figure 7: $n = 3, k = 3$

For graphs $g_{12}, g_{13} \in \mathfrak{G}_{3,3}$ we have:

$$[g_{12}] = [g_{13}] = \{1, 1, 1, 1, 2\}$$

$$\Psi(g_{12}) = \Psi(g_{13}) = \langle 1, 4, 1, 0 \rangle$$

$$\omega(g_{12}) = \omega(g_{13}) = \frac{[(3-0)!]^1 [(3-1)!]^4}{1! 1! 1! 1! 2!} = \frac{6^1 \cdot 2^4}{2} = 48$$

For graph $g_{14} \in \mathfrak{G}_{3,3}$ we have:

$$[g_{14}] = \{1, 1, 1, 1, 1, 1\}$$

$$\Psi(g_{14}) = \langle 2, 2, 2, 0 \rangle$$

$$\omega(g_{14}) = \frac{[(3-0)!]^2 [(3-1)!]^2}{1! 1! 1! 1! 1! 1!} = 6^2 \cdot 2^2 = 144$$

For graphs $g_{15}, g_{16} \in \mathfrak{G}_{3,3}$ we have:

$$[g_{15}] = [g_{16}] = \{1, 2, 3\}$$

$$\Psi(g_{15}) = \Psi(g_{16}) = \langle 2, 3, 0, 1 \rangle$$

$$\omega(g_{15}) = \omega(g_{16}) = \frac{[(3-0)!]^2 [(3-1)!]^3}{1! 2! 3!} = \frac{6^2 \cdot 2^3}{2 \cdot 6} = 24$$

Then for the set $\mathfrak{G}_{3,3}$ we get:

$$\theta(3, 3) = \sum_{g \in \mathfrak{G}_{3,3}} \omega(g) = 64 + 48 + 48 + 144 + 24 + 24 = 352. \quad (17)$$

3.2.4 $k = 4$

When $n = 3$ and $k = 4$ the set $\mathfrak{G}_{3,4} = \{g_{17}, g_{18}, g_{19}, g_{20}, g_{21}, g_{22}, g_{23}\}$ consists of seven bipartite graphs, which are shown in Figure 8.

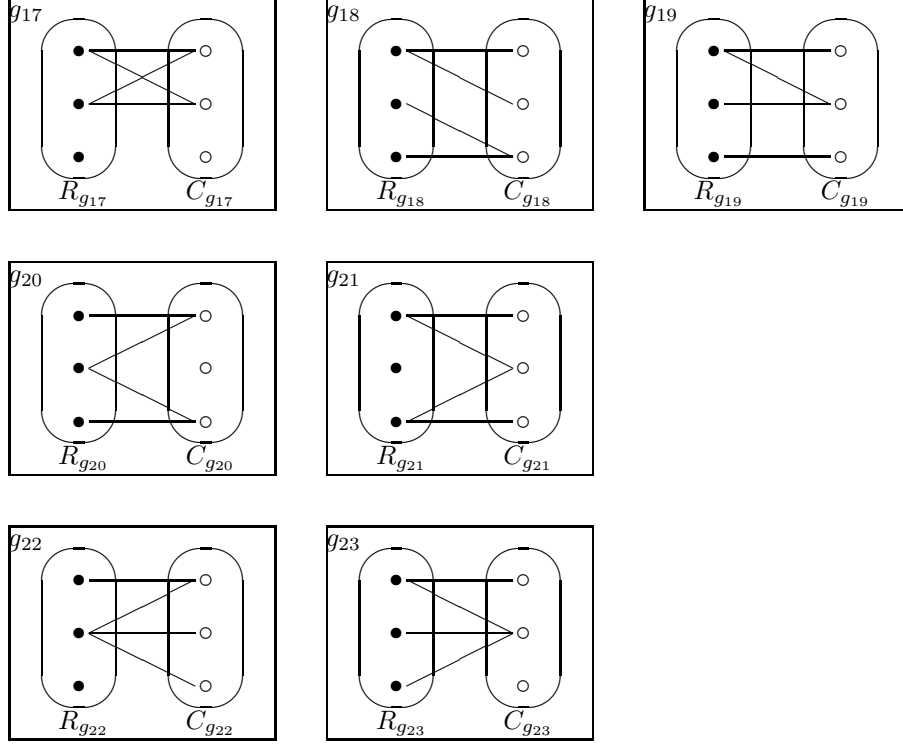


Figure 8: $n = 3, k = 4$

For graph $g_{17} \in \mathfrak{G}_{3,4}$ we have:

$$[g_{17}] = \{1, 1, 2, 2\}$$

$$\Psi(g_{17}) = \langle 2, 0, 4, 0 \rangle$$

$$\omega(g_{17}) = \frac{[(3-0)!]^2 [(3-1)!]^0}{1! 1! 2! 2!} = \frac{6^2 \cdot 2^0}{2^2} = 9$$

For graph $g_{18} \in \mathfrak{G}_{3,4}$ we have:

$$[g_{18}] = \{1, 1, 2, 2\}$$

$$\Psi(g_{18}) = \langle 0, 4, 2, 0 \rangle$$

$$\omega(g_{18}) = \frac{[(3-0)!]^0 [(3-1)!]^4}{1! 1! 2! 2!} = \frac{6^0 \cdot 2^4}{2^2} = 4$$

For graph $g_{19} \in \mathfrak{G}_{3,4}$ we have:

$$[g_{19}] = \{1, 1, 1, 1, 1, 1\}$$

$$\Psi(g_{19}) = \langle 0, 4, 2, 0 \rangle$$

$$\omega(g_{19}) = \frac{[(3-0)!]^0 [(3-1)!]^4}{1! 1! 1! 1! 1! 1!} = 6^0 \cdot 2^4 = 16$$

For graphs $g_{20} \in \mathfrak{G}_{3,4}$ and $g_{21} \in \mathfrak{G}_{3,4}$ we have:

$$[g_{20}] = [g_{21}] = \{1, 1, 1, 1, 1, 1\}$$

$$\Psi(g_{20}) = \Psi(g_{21}) = \langle 1, 2, 3, 0 \rangle$$

$$\omega(g_{20}) = \omega(g_{21}) = \frac{[(3-0)!]^1 [(3-1)!]^2}{1! 1! 1! 1! 1! 1!} = 6^1 \cdot 2^2 = 24$$

For graphs $g_{22} \in \mathfrak{G}_{3,4}$ and $g_{23} \in \mathfrak{G}_{3,4}$ we have:

$$[g_{22}] = [g_{23}] = \{1, 1, 1, 1, 2\}$$

$$\Psi(g_{22}) = \Psi(g_{23}) = \langle 1, 3, 1, 1 \rangle$$

$$\omega(g_{22}) = \omega(g_{23}) = \frac{[(3-0)!]^1 [(3-1)!]^3}{1! 1! 1! 1! 2!} = \frac{6^1 \cdot 2^3}{2} = 24$$

Then we get:

$$\theta(3, 4) = \sum_{g \in \mathfrak{G}_{3,4}} \omega(g) = 9 + 4 + 16 + 24 + 24 + 24 + 24 = 125. \quad (18)$$

3.2.5 $k = 5$

When $n = 3$ and $k = 5$ the set $\mathfrak{G}_{3,5}$ consists of seven bipartite graphs $g_{24} \div g_{30}$, which are shown in Figure 9.

For graph $g_{24} \in \mathfrak{G}_{3,5}$ we have:

$$[g_{24}] = \{1, 1, 2, 2\}$$

$$\Psi(g_{24}) = \langle 0, 4, 0, 2 \rangle$$

$$\omega(g_{18}) = \frac{[(3-0)!]^0 [(3-1)!]^4}{1! 1! 2! 2!} = \frac{6^0 \cdot 2^4}{2^2} = 4$$

For graph $g_{25} \in \mathfrak{G}_{3,5}$ we have:

$$[g_{25}] = \{1, 1, 2, 2\}$$

$$\Psi(g_{25}) = \langle 0, 2, 4, 0 \rangle$$

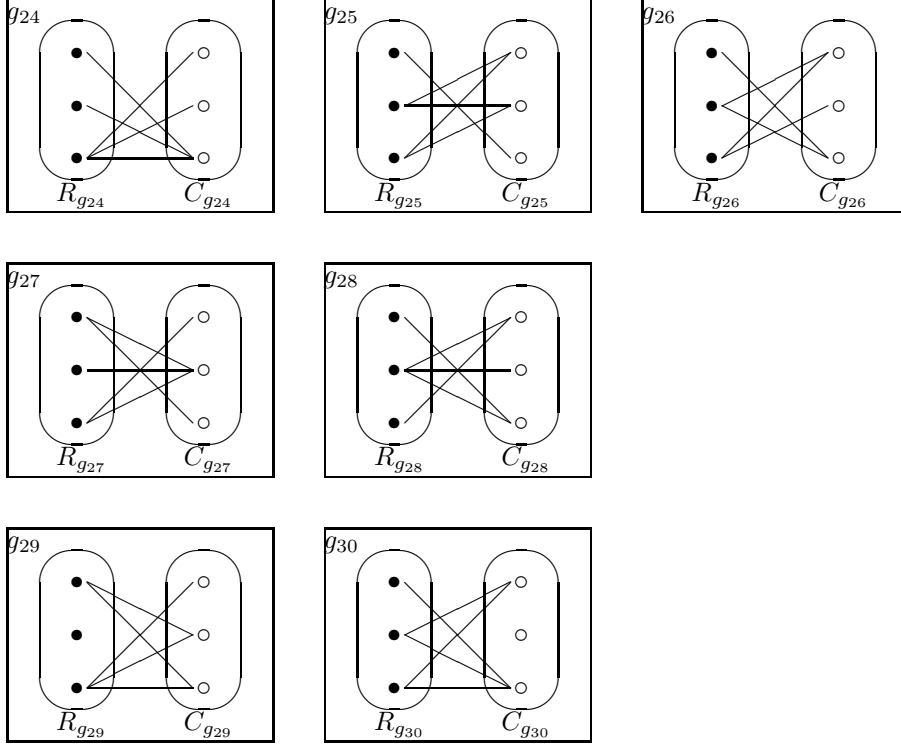


Figure 9: $n = 3, k = 5$

$$\omega(g_{18}) = \frac{[(3-0)!]^0 [(3-1)!]^2}{1! 1! 2! 2!} = \frac{6^0 \cdot 2^2}{2^2} = 1$$

For graph $g_{26} \in \mathfrak{G}_{3,5}$ we have:

$$[g_{26}] = \{1, 1, 1, 1, 1, 1\}$$

$$\Psi(g_{26}) = \langle 0, 2, 4, 0 \rangle$$

$$\omega(g_{26}) = \frac{[(3-0)!]^0 [(3-1)!]^2}{1! 1! 1! 1! 1! 1!} = 6^0 \cdot 2^2 = 4$$

For graphs $g_{27} \in \mathfrak{G}_{3,5}$ and $g_{28} \in \mathfrak{G}_{3,5}$ we have:

$$[g_{27}] = [g_{28}] = \{1, 1, 1, 1, 1, 1\}$$

$$\Psi(g_{27}) = \Psi(g_{28}) = \langle 0, 3, 2, 1 \rangle$$

$$\omega(g_{27}) = \omega(g_{28}) = \frac{[(3-0)!]^0 [(3-1)!]^3}{1! 1! 1! 1! 1! 1!} = 6^0 \cdot 2^3 = 8$$

For graphs $g_{29} \in \mathfrak{G}_{3,5}$ and $g_{30} \in \mathfrak{G}_{3,5}$ we have:

$$[g_{29}] = [g_{30}] = \{1, 1, 1, 1, 2\}$$

$$\Psi(g_{29}) = \Psi(g_{30}) = \langle 1, 1, 3, 1 \rangle$$

$$\omega(g_{29}) = \omega(g_{30}) = \frac{[(3-0)!]^1 [(3-1)!]^1}{1! 1! 1! 1! 2!} = \frac{6^1 \cdot 2^1}{2} = 6$$

Then we get:

$$\theta(3, 5) = \sum_{g \in \mathfrak{G}_{3,5}} \omega(g) = 4 + 1 + 4 + 8 + 8 + 6 + 6 = 37. \quad (19)$$

3.2.6 $k = 6$

When $n = 3$ and $k = 6$ the set $\mathfrak{G}_{3,6} = \{g_{31}, g_{32}, g_{33}, g_{34}, g_{35}, g_{36}\}$ consists of six bipartite graphs, which are shown in Figure 10.

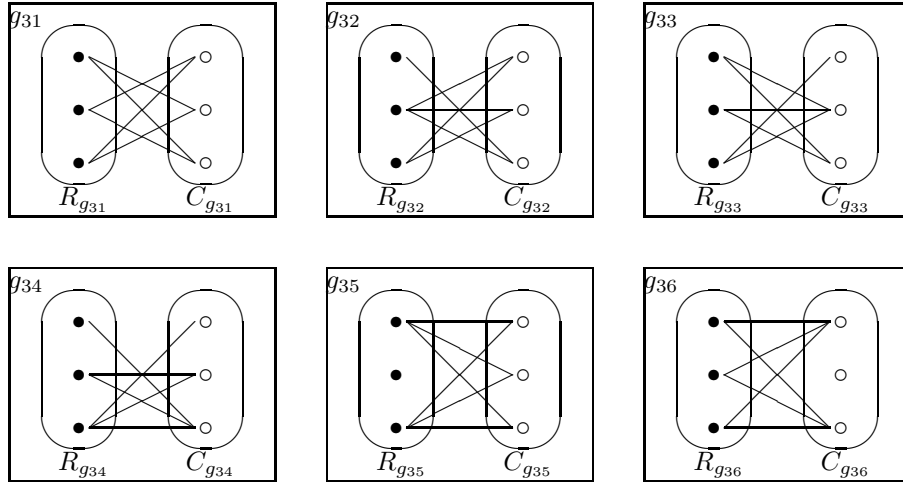


Figure 10: $n = 3, k = 6$

For graph $g_{31} \in \mathfrak{G}_{3,6}$ we have:

$$[g_{31}] = \{1, 1, 1, 1, 1, 1\}$$

$$\Psi(g_{31}) = \langle 0, 0, 6, 0 \rangle$$

$$\omega(g_{31}) = \frac{[(3-0)!]^0 [(3-1)!]^0}{1! 1! 1! 1! 1! 1!} = 1$$

For graphs $g_{32} \in \mathfrak{G}_{3,6}$ and $g_{33} \in \mathfrak{G}_{3,6}$ we have:

$$[g_{32}] = [g_{33}] = \{1, 1, 1, 1, 2\}$$

$$\Psi(g_{32}) = \Psi(g_{33}) = \langle 0, 1, 4, 1 \rangle$$

$$\omega(g_{32}) = \omega(g_{33}) = \frac{[(3-0)!]^0 [(3-1)!]^1}{1! 1! 1! 1! 2!} = \frac{6^0 \cdot 2^1}{2} = 1$$

For graph $g_{34} \in \mathfrak{G}_{3,6}$ we have:

$$[g_{34}] = \{1, 1, 1, 1, 1, 1\}$$

$$\Psi(g_{34}) = \langle 0, 2, 2, 2 \rangle$$

$$\omega(g_{34}) = \frac{[(3-0)!]^0 [(3-1)!]^2}{1! 1! 1! 1! 1! 1!} = \frac{6^0 \cdot 2^2}{1} = 4$$

For graphs $g_{35} \in \mathfrak{G}_{3,6}$ and $g_{36} \in \mathfrak{G}_{3,6}$ we have:

$$[g_{35}] = [g_{36}] = \{1, 2, 3\}$$

$$\Psi(g_{35}) = \Psi(g_{36}) = \langle 1, 0, 3, 2 \rangle$$

$$\omega(g_{35}) = \omega(g_{36}) = \frac{[(3-0)!]^1 [(3-1)!]^0}{1! 2! 3!} = \frac{6^1 \cdot 2^0}{2 \cdot 6} = \frac{1}{2}$$

Then for the set $\mathfrak{G}_{3,6}$ we get:

$$\theta(3, 6) = \sum_{g \in \mathfrak{G}_{3,6}} \omega(g) = 1 + 1 + 1 + 4 + \frac{1}{2} + \frac{1}{2} = 8 \quad (20)$$

3.2.7 $k = 7$

When $n = 3$ and $k = 7$ the set $\mathfrak{G}_{3,7} = \{g_{37}, g_{38}, g_{39}\}$ consists of three bipartite graphs, which are shown in Figure 11.

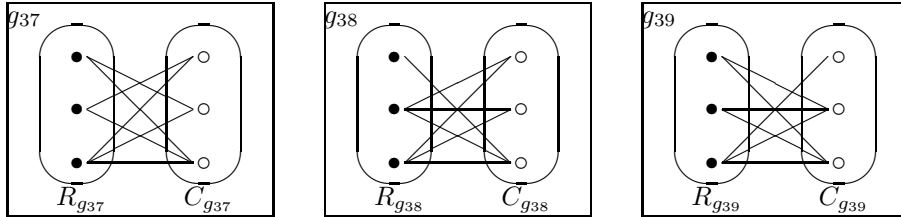


Figure 11: $n = 3, k = 7$

For graph $g_{37} \in \mathfrak{G}_{3,7}$ it is true:

$$[g_{37}] = \{1, 1, 1, 1, 1, 1\}$$

$$\Psi(g_{37}) = \langle 0, 0, 4, 2 \rangle$$

$$\omega(g_{37}) = \frac{[(3-0)!]^0 [(3-1)!]^0}{1! 1! 1! 1! 1! 1!} = \frac{6^0 \cdot 2^0}{1} = 1$$

For graphs $g_{38} \in \mathfrak{G}_{3,7}$ and $g_{39} \in \mathfrak{G}_{3,7}$ we get:

$$[g_{38}] = [g_{39}] = \{1, 1, 2, 2\}$$

$$\Psi(g_{38}) = \Psi(g_{39}) = \langle 0, 1, 2, 3 \rangle$$

$$\omega(g_{38}) = \omega(g_{39}) = \frac{[(3-0)!]^0 [(3-1)!]^1}{1! 1! 2! 2!} = \frac{6^0 \cdot 2^1}{2^2} = \frac{1}{2}$$

Then for the set $\mathfrak{G}_{3,7}$ we get:

$$\theta(3, 7) = \sum_{g \in \mathfrak{G}_{3,7}} \omega(g) = 1 + \frac{1}{2} + \frac{1}{2} = 2 \quad (21)$$

3.2.8 $k = 8$

Graph g_{40} , which is displayed in Figure 12 is the only bipartite graph belonging to the set $\mathfrak{G}_{3,8}$ in the case $n = 3$ and $k = 8$.

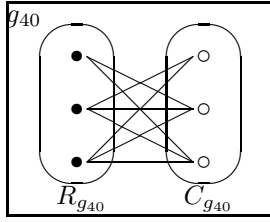


Figure 12: $n = 3, k = 8$

For graph $g_{40} \in \mathfrak{G}_{3,8}$ it is true:

$$[g_{40}] = \{1, 1, 2, 2\}$$

$$\Psi(g_{40}) = \langle 0, 0, 2, 4 \rangle$$

$$\omega(g_{40}) = \frac{[(3-0)!]^0 [(3-1)!]^0}{1! 1! 2! 2!} = \frac{6^0 \cdot 2^0}{2^2} = \frac{1}{4}$$

Therefore:

$$\theta(3, 8) = \sum_{g \in \mathfrak{G}_{3,8}} \omega(g) = \frac{1}{4} \quad (22)$$

3.2.9 $k = 9$

When $n = 3$ and $k = 9$ there is only one graph and this is the complete bipartite graph g_{41} which is shown in Figure 13.

For graph g_{41} is true:

$$[g_{41}] = \{3, 3\}$$

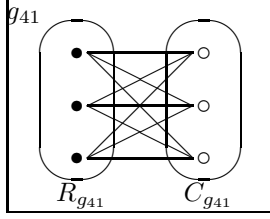


Figure 13: $n = 3, k = 9$

$$\Psi(g_{41}) = \langle 0, 0, 0, 6 \rangle$$

$$\omega(g_{41}) = \frac{[(3-0)!]^0 [(3-1)!]^0}{3! 3!} = \frac{6^0 \cdot 2^0}{6^2} = \frac{1}{36}$$

Therefore

$$\theta(3, 9) = \sum_{g \in \mathfrak{S}_{3,9}} \omega(g) = \frac{1}{36} \quad (23)$$

Having in mind the formula (8) and formulas (15) \div (23) for the number D_9 of all ordered pairs disjoint S-permutation matrices in $n = 3$ we finally get:

$$\begin{aligned} D_9 &= (3!)^{12} + (3!)^8 \left[\sum_{k=1}^9 (-1)^k \theta(n, k) \right] = \quad (24) \\ &= 2\,176\,782\,336 + 1\,679\,616 \left(-1296 + 1008 - 352 + 125 - 37 + 8 - 2 + \frac{1}{4} - \frac{1}{36} \right) = \\ &= 1\,260\,085\,248. \end{aligned}$$

The number d_9 of all non-ordered pairs disjoint matrices from Σ_9 is equal to

$$d_9 = \frac{1}{2} D_9 = 630\,042\,624 \quad (25)$$

3.3 On a combinatorial problem of graph theory related to the number of Sudoku matrices

Problem 1 *Let $n \geq 2$ is a natural number and let G be a simple graph having $(n!)^{2n}$ vertices. Let each vertex of G be identified with an element of the set Σ_{n^2} of all $n^2 \times n^2$ S-permutation matrices. Two vertices are connected by an edge if and only if the corresponding matrices are disjoint. The problem is to find the number of all complete subgraphs of G having n^2 vertices:*

Note that the number of edges in graph G is equal to d_{n^2} and can be calculated using formula (2) and formula (3) (respectively formulas (6), (7), (8) and (3)).

Denote by z_n the solution of the Problem 1 and let σ_n is the number of all $n^2 \times n^2$ Sudoku matrices. Then according to Proposition 1 and the method of construction of the graph G , it follows that the next equality is valid:

$$z_n = \frac{\sigma_n}{(n^2)!} \quad (26)$$

We do not know a general formula for finding the number of all $n^2 \times n^2$ Sudoku matrices for each natural number $n \geq 2$ and we consider that this is an open combinatorial problem. Only some special cases are known. For example in $n = 2$ it is known that $\sigma_2 = 288$ [7]. Then according to formula (26) we get:

$$z_2 = \frac{\sigma_2}{4!} = \frac{288}{24} = 12$$

In [4] it has been shown that in $n = 3$ there are exactly

$$\begin{aligned} \sigma_3 &= 6\,670\,903\,752\,021\,072\,936\,960 = \\ &= 9! \times 72^2 \times 2^7 \times 27\,704\,267\,971 = \\ &2^{20} \times 3^8 \times 5^1 \times 7^1 \times 27\,704\,267\,971^1 \sim 6.671 \times 10^{21} \end{aligned}$$

number of Sudoku matrices. Then according to formula (26) we get:

$$z_3 = \frac{\sigma_3}{9!} = \frac{6\,670\,903\,752\,021\,072\,936\,960}{362\,880} = 18\,383\,222\,420\,692\,992$$

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